A note on the boundary layer equations with linear slip boundary condition

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Abstract

In a recent article the authors study the effect of replacing the standard no-slip boundary condition with a nonlinear Navier boundary condition for the boundary layer equations. The resulting equations contain an arbitrary index parameter, denoted by \( n \), and it is found that the case \( n = 1 \) corresponding to linear Navier boundary condition must be excluded. In this article the authors remedy this situation and show that the case \( n = 1 \) corresponds to a particular similarity solution, not included in the previous work by the authors.

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1. Introduction

In a previous article Matthews and Hill [1] study the classical boundary layer similarity solutions studied in detail by Falkner and Skan [2]. The aim of this analysis is to determine the effect of replacing the standard no-slip boundary condition of fluid mechanics applying for the so-called Falkner–Skan solutions, with a boundary condition that allows some degree of tangential fluid slip. The particular slip boundary condition is a variant of the linear Navier boundary condition, first proposed independently by Navier [3] and Maxwell [4], which postulates that the component of the fluid velocity tangential to the surface is assumed to be proportional to the tangential stress and that the constant of proportionality is called the slip length. This variation is motivated by the experimental work of Choi et al. [5], who suggest that a nonlinear Navier boundary condition with the tangential stress raised to a positive power \( n \), provides an improved slip model for microfluidic applications. In the previous work Matthews and Hill [1] find that for the similarity solution the case \( n = 1 \), which corresponds to the linear Navier boundary condition, must be excluded. The present work reveals that the case \( n = 1 \) in fact corresponds to a special case of a general similarity reduction.

In the following section the laminar boundary layer equations are described. In the subsequent sections the classical similarity reductions of the boundary layer equations are derived. The special case overlooked in Matthews and Hill [1] is discussed, and the resulting ordinary differential equation is solved numerically using MAPLE’s built-in boundary value problem solvers. Finally, we present a discussion of the results and we make some concluding remarks.

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2. Boundary layer equations

With the assumption that viscosity is negligible in the bulk flow but important near a solid boundary, the steady two-dimensional boundary layer equations may be written in terms of a stream function $\psi (x, y)$ defined via the following relation (Schlichting [6])

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \tilde{v}_x \frac{d\tilde{v}_x}{dx} + \frac{\partial^3 \psi}{\partial y^3}, \quad -\infty < x < \infty, \; y > 0,$$

(2.1)

where

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x},$$

(2.2)

and $x$ is measured parallel and $y$ is measured perpendicular to a solid boundary located along $y = 0$. Finally, $\tilde{v}_x$ is an assumed given external inviscid velocity field such that $v_x \rightarrow \tilde{v}_x$ as $y \rightarrow \infty$. The above partial differential equation is solved with suitable boundary conditions imposed at $y = 0$ and matching with the inviscid solution as $y \rightarrow \infty$. The specific boundary conditions imposed in this study are

$$y = 0: |v_x| = \ell \left| \frac{\partial v_x}{\partial y} \right|^n; \quad v_y = 0, \quad \text{and} \quad y \rightarrow \infty: v_x \equiv \tilde{v}_x = x,$$

(2.3)

where the constant $\ell > 0$ is the slip length which has been scaled with $N_{Re}^{1/2}$ where $N_{Re}$ is the Reynolds number. Here we set $n = 1$ to make the boundary condition linear. In terms of the stream function these are

$$y = 0: \left| \frac{\partial \psi}{\partial y} \right| = \ell \left| \frac{\partial^2 \psi}{\partial y^2} \right|^n; \quad \frac{\partial \psi}{\partial x} = 0, \quad \text{and} \quad y \rightarrow \infty: \frac{\partial \psi}{\partial y} = x.$$

(2.4)

2.1. Classical symmetry reductions

The Lie-group method of infinitesimal transformations is used for finding the symmetries of the boundary layer equations, and the general method is explained elsewhere [7]. The inviscid velocity terms in the governing partial differential equation may be eliminated by taking the $y$ derivative of both sides, so that

$$\frac{\partial \psi}{\partial y} \frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial \psi}{\partial x} \frac{\partial^3 \psi}{\partial y^2 \partial x} = \frac{\partial^4 \psi}{\partial y^4}.$$

(2.5)

Using the MAPLE package DESOLV [8], it is found that this partial differential equation has the following Lie symmetry infinitesimals

$$X = (C_1 + C_4) x + C_2, \quad Y = C_4 y + f (x), \quad \Psi = C_1 \psi + C_3,$$

(2.6)

where $C_i \; (i = 1, 2, 3, 4)$ denote arbitrary constants and $f (x)$ is a sufficiently differentiable arbitrary function. In Matthews and Hill [11] the cases $C_4 = 1$ and $C_4 = C_1$ are studied in detail. Here we set $C_4 = 0 = C_2 = C_3 = f (x)$, then the group invariant solution of the boundary layer equations is of the form

$$\psi (x, y) = x F (y).$$

(2.7)

Substituting into the boundary layer equation yields an ordinary differential equation for $F (y)$

$$\frac{d^3 F}{dy^3} + \frac{d}{dy} \left( \frac{F}{dF} \frac{dF}{dy} \right) - 2 \left( \frac{dF}{dy} \right)^2 + 1 = 0,$$

(2.8)

which may alternatively be written as

$$\frac{d^3 F}{dy^3} + F \frac{d^2 F}{dy^2} - \left( \frac{dF}{dy} \right)^2 + 1 = 0.$$

(2.9)
This must be solved subject to

\[ y = 0; \quad \frac{dF}{dy} = \ell \left| \frac{d^2F}{dy^2} \right|; \quad F = 0, \quad \text{and} \quad y \to \infty: \frac{dF}{dy} \to 1, \tag{2.10} \]

and we also have

\[ \frac{v_x}{\tilde{v}_x} = \frac{dF}{dy}, \quad x \frac{v_y}{\tilde{v}_x} = -F. \tag{2.11} \]

As demonstrated by Falkner and Skan [2] the family of inviscid flows with \( \tilde{v}_x \sim x \) corresponds to the flow past a wedge with angle \( \pi \) (that is, stagnation point flow) and as such the analysis applies to the boundary layer formed near the wedge. This same analogy may be applied in this analysis, since in the inviscid limit the linear Navier slip boundary condition is dropped.

Eq. (2.9) with the boundary conditions given by Eq. (2.10) are solved numerically for the cases \( \ell = 0, 0.1, 0.5 \) and \( 1 \) using MAPLE. The details of the numerical method used are given in the appendix of Matthews and Hill [1]. The profiles for the \( x \) and \( y \) components of velocity given by Eq. (2.11) are illustrated in Figs. 1 and 2.
3. Discussion and concluding remarks

In this study the classical laminar boundary layer equations are solved with a linear slip boundary condition. Solutions are obtained via a special case of the classical Lie point symmetry reductions, and the resulting ordinary differential equation is solved numerically.

For the boundary layer analysis, the numerical solution demonstrates that as the slip length increases the rate of change of the tangential velocity through the boundary layer decreases, since the velocity at the solid surface is no longer zero and slips with a velocity which increases as $\ell$ increases. However, there is very little effect on the normal velocity through the boundary layer. The velocity components are increased in magnitude as the slip length is increased, which is to be expected.

In conjunction with Matthews and Hill [1] there still remains one open question, which is to determine the flow past a flat plate with the proposed form of slip boundary condition. This somewhat unexpected outcome is an artifact of the assumed similarity solution, which does not lend itself to this problem. This could possibly indicate that the scaling arguments used to obtain the boundary layer equations must be altered for a boundary layer involving tangential fluid slip at solid interfaces.

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