INVERSES OF VANDERMONDE MATRICES

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1. Introduction. In a recent paper [1], one finds explicit formulas for the derivatives of a polynomial \( y = f(x) \) of degree \( n \) in terms of its values \( y_i = f(x_i) \) at \( n+1 \) points defined by \( x_i = x_0 + ih, \ (i=0, \cdots, n) \). These results are used here to invert the Vandermonde matrix

\[
V(x_1, \cdots, x_n) = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
\vdots & \vdots & \cdots & \vdots \\
x_{n-1} & x_{n-1} & \cdots & x_{n-1}
\end{bmatrix},
\]

where the \( x_i \) are distinct, different from zero, but otherwise arbitrary.

The paper is in three parts. In the first we outline the results from [1] required later. In the second, these formulas are applied to the special Vandermonde matrix

\[
V_{n+1}(x_0, h) = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_0 & x_0 + h & \cdots & x_0 + nh \\
\vdots & \vdots & \cdots & \vdots \\
x_0^n & (x_0 + h)^n & \cdots & (x_0 + nh)^n
\end{bmatrix};
\]

and the elements of \( V_{n+1}^{-1}(x_0, h) \) are obtained in terms of Stirling numbers. Finally, the methods of [1] are extended in such a way that the elements of \( V^{-1}(x_1, \cdots, x_n) \) can be expressed in terms of the elementary symmetric functions of the \( x_i \)'s. This last result is offered as an alternative to the derivation one would obtain from the classical formula for the values of Vandermonde determinants with missing powers ([2], p. 99).

2. Preliminary results. Let \( y = f(x) \) be a polynomial of degree \( n \), and \( y_i = f(x_i), \ (i=0, 1, \cdots, n) \), where \( x_i = x_0 + ih \). It was shown in [1] that if we write

\[
(1) \quad h^k f^{(k)}(x) = \sum_{i=0}^{n} A_{mi} y_i,
\]

then

\[
(2) \quad A_{mi} = \sum_{j=k}^{n} \frac{(-1)^{i+j} \binom{j}{i} C_{mj}^k}{j!},
\]

where

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\[
\binom{j}{i} = \begin{cases} 
1 & \text{if } i = 0 \\
0 & \text{if } i > j,
\end{cases}
\]

(3)

\[
C_{mj}^k = \sum_{r=k}^i p_k(r) S_r^{m-r},
\]

\(m\) is any real number, and \(x = x_0 + mh\). In the above, \(p_k(r)\) denotes the factorial polynomial of degree \(k\), and the \(S_k^r\) are the Stirling numbers of the first kind. Thus, we have \(p_k(x) = \sum_{j=1}^k S_k^j x^j\). A wide variety of classical numerical differentiation formulas are special cases of (1).

It was shown further that these results enable one to invert the Vandermonde matrix

\[
M(m) = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
-m & 1 - m & \cdots & n - m \\
(-m)^n & (1 - m)^n & \cdots & (n - m)^n
\end{bmatrix}.
\]

If we write \(M^{-1}(m) = \{a_{\lambda,\mu}^m\}\), \((\lambda, \mu = 1, \cdots, n+1)\), then

(4)

\[
a_{\lambda,\mu}^m = \frac{A_{m,\lambda-1}^{\mu-1}}{\mu!},
\]

where \(A_{m,\lambda-1}^{\mu-1}\) is given by (2).

3. Vandermonde Matrices for equally spaced points. It is easy to show that

\[
M(m) = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
-mh & -mh + h & \cdots & -mh + nh \\
(-mh)^n & (-mh + h)^n & \cdots & (-mh + nh)^n
\end{bmatrix}.
\]

If we write \(x_0 = -mh\), it follows that

\[
V_{n+1}(x_0, h) = \begin{bmatrix}
1 & 1 & 1 \\
x_0 & x_0 + h & \cdots & x_0 + nh \\
x_0^n & (x_0 + h)^n & \cdots & (x_0 + nh)^n
\end{bmatrix}.
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & h & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & h^n
\end{bmatrix} M(-x_0/h).
\]
and so

\[
V_{n+1}^{-1}(x_0, h) = M^{-1}(-x_0/h)
\]

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & h^{-1} & 0 & \cdots & 0 \\
& & \ddots & & \ddots \\
0 & 0 & 0 & \cdots & h^{-n}
\end{bmatrix}.
\]

Thus, by (4) the element in the \( \lambda \)th row and \( \mu \)th column of \( V_{n+1}^{-1}(x_0, h) \) is precisely

\[
\nu_{\lambda \mu} = \frac{1}{h^{\mu-1}(\mu - 1)!} A_{-x_0/h, \lambda - 1}^{\mu-1}.
\]

It is interesting to note that even though \( V_{n+1}^{-1}(1, 1) \) can be obtained directly from the above, a slight modification enables one to express the elements of this inverse very compactly. For simplicity of notation, we apply the method to obtain \( V_n^{-1}(1, 1) \).

Let us write

\[
M(0) = \begin{bmatrix} 1 & Q_{1n} \\ R_{n1} & P_{nn} \end{bmatrix}, \quad M^{-1}(0) = \begin{bmatrix} b_{11} & S_{1n} \\ T_{n1} & U_{nn} \end{bmatrix},
\]

where \( Q_{1n} = (1, \cdots, 1) \), \( R_{n1} \) is a column vector of zeros, and

\[
P_{nn} = \begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 4 & \cdots & n^2 \\ & \ddots & \ddots & \ddots \\ 1 & 2^n & \cdots & n^n \end{bmatrix}.
\]

It suffices to invert the matrix \( P_{nn} \), since its columns are scalar multiples of those of \( V_n(1, 1) \). Now

\[
M(0)M^{-1}(0) = \begin{bmatrix} b_{11} + Q_{1n}T_{n1} & S_{1n} + Q_{1n}U_{nn} \\ R_{n1}b_{11} + P_{nn}T_{n1} & R_{n1}S_{1n} + P_{nn}U_{nn} \end{bmatrix} = I,
\]

and so \( I_n = R_{n1}S_{1n} + P_{nn}U_{nn} = P_{nn}U_{nn} \). Hence \( U_{nn} = P_{nn}^{-1} \). Denote the elements of \( U_{nn} \) by \( \mu_{ik} \). Since \( b_{11} \) contains only a single element, it follows that \( \mu_{ik} = a_{i+1,k+1}^0 \), and in turn, by (4), \( \mu_{ik} = A_{0,i}^k/k! \). We have from (2) that

\[
A_{0,i}^k = \sum_{j=k}^n \frac{(-1)^{j+i} \binom{j}{i} C_{0j}^k}{j!} \quad (i, k = 1, \cdots, n).
\]

From (3) \( C_{0j}^k = \rho_k(k)S_j^k = k!S_j^k \). Finally, we have
The inverse of the general Vandermonde matrix. Let \( x_0, x_1, \ldots, x_n \) be the given distinct numbers. The polynomial \( y = y(x) \) of degree \( n \) assuming \( n+1 \) arbitrary values \( y_i = y(x_i), \ i = 0, \ldots, n \), can be written, by Lagrange's interpolation formula, as

\[
y = \sum_{i=0}^{n} A_{x_i} y_i.
\]

If we write the \( k \)th derivative of \( y(x) \) as

\[
y^{(k)} = \sum_{i=0}^{n} A_{x_i}^{k} y_i \quad (k = 1, \ldots, n),
\]

where

\[
A_{x_i}^{k} = \frac{d^k}{dx^k} A_{x_i}.
\]

and set \( x = x_0 = 0 \), we have
In the following, we obtain explicit expressions for the $A_{0,t}^k$, and then show that the $A_{0,t}^k$ satisfy systems of linear equations having the same coefficient matrix. Thus, we obtain the inverse of this matrix and, in turn, the inverse of $V(x_1, \cdots, x_n)$ in terms of the $A_{0,t}^k$.

The functions $A_{xt}$, as given by the Lagrange interpolation formula, are

$$A_{xt} = \frac{1}{p_{n+1}'(x_t)} \cdot \frac{p_{n+1}(x)}{(x - x_t)}$$

and so

$$A_{xt} = \frac{1}{p_{n+1}'(x_t)} \left[ x^n - \sigma_{1,n-1} x^{n-1} + \sigma_{2,n-1} x^{n-2} - \cdots + (-1)^{n-2} \sigma_{n-2,n-1} x^2 + (-1)^{n-1} \sigma_{n-1,n-1} x \right],$$

where $\sigma_{j,n-1}$ is the sum of all products of $j$ of the numbers $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ without permutations or repetitions ($\sigma_{0,n-1} = 1$). If we differentiate the above $k$ times, set $x=0$, and notice that

$$p_{n+1}'(x_t) = \prod_{j=0}^{n} (x_t - x_j),$$

where the dash indicates that $i \neq j$, we obtain

$$A_{0,t}^k = \frac{(-1)^{n-k} \cdot k!}{\sigma_{n-k,n-1}} \cdot \prod_{j=0}^{n} (x_t - x_j)$$

In order to exhibit the linear systems mentioned above, we expand $y(x)$ about $x_0 = 0$, and substitute $x = x_t$ to get

$$y_t = \sum_{\mu=0}^{n} \frac{1}{\mu!} y_0^{(\mu)} x_t^{\mu}.$$  

* More generally, if we do not substitute $x = 0$ after the above differentiation, and substitute the result in (7), we obtain

$$y^{(k)}(x) = \sum_{i=0}^{n} y_i \sum_{j=k}^{n} \frac{(-1)^{n-j} \cdot j! \sigma_{n-j,n-1} x^{j-1} \cdots (j - k + 1) x^{j-k}}{p_{n+1}'(x_i)},$$

which is a formula for an arbitrary derivative of $y(x)$ at an arbitrary $x$. 

(8) 

$$y_0^{(k)} = \sum_{i=0}^{n} A_{0,i}^k y_i.$$
Inserting this into (8) and rearranging, we get

\[ y^{(k)}_0 = \sum_{\mu=0}^{n} \frac{1}{\mu!} y^{(\mu)}_0 \sum_{i=0}^{n} x_i^\mu A_{0i}^k \quad (k = 1, \ldots, n). \]

Since these are identities in \( y^{(k)}_0 \), we must have

\[ \sum_{i=0}^{n} x_i^\mu A_{0i}^k = \delta_{\mu k} k! \quad (\mu = 0, 1, \ldots, n), \]

where \( \delta_{\mu k} \) is the Kronecker delta. Since \( x_0 = 0 \), it follows that

\[ \sum_{i=1}^{n} x_i^\mu A_{0i}^k = \delta_{\mu k} \cdot k! \quad (\mu = 1, \ldots, n). \]

For each \( k \) \((k = 1, \ldots, n)\), this is a system of \( n \) linear equations in the unknowns \( A_{01}^k, A_{02}^k, \ldots, A_{0n}^k \). These \( n \) systems can be combined into the matrix equation

\[
\begin{bmatrix}
1! & 0 & 0 & \cdots & 0 \\
0 & 2! & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & n! \\
\end{bmatrix}
\begin{bmatrix}
x_1^i \\
x_2^i \\
\vdots \\
x_n^i \\
\end{bmatrix} =
\begin{bmatrix}
x_1^i \\
x_2^i \\
\vdots \\
x_n^i \\
\end{bmatrix} = \{ A_{0i}^k \} = \{ x_k^i \}^{-1} \cdot D(n). 
\]

where \( A_{0i}^k \) is the element of \( \{ A_{0i}^k \} \) in the \( i \)th row and \( k \)th column, and similarly for the element \( x_k^i \) of \( \{ x_k^i \} \). (In \( x_k^i \) the \( i \) denotes an actual exponent.) Denoting the right member of the above equation by \( D(n) \), we may write \( \{ A_{0i}^k \} = \{ x_k^i \}^{-1} \cdot D(n) \). Thus, the element \( b_{\lambda \mu} \) in the \( \lambda \)th row and \( \mu \)th column of \( \{ x_k^i \}^{-1} \) \((\lambda, \mu = 1, \ldots, n)\) is given by

\[ b_{\lambda \mu} = \frac{1}{\mu!} A_{\lambda \lambda}^\mu, \]

where the \( A_{\lambda \alpha}^\mu \) may be obtained from (9).

Finally, if \( V^{-1}(x_1, \ldots, x_n) = \{ v_{\lambda \mu} \} \), a method similar to that used in Section 3 yields \( v_{\lambda \mu} = x_\lambda b_{\lambda \mu} \), \((\lambda, \mu = 1, \ldots, n)\), which, together with (10), gives an explicit representation for the inverse of the general Vandermonde matrix.

References